Parametrization of the Driven Betatron Oscillation

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The AC dipole is a short dipole magnet to excite transverse motion of a beam to diagnose a synchrotron. Instead of giving a single pulse like a conventional kicker/pinger magnet, it drives the beam with its sinusoidally time varying field. The transverse motions excited by an AC dipole (driven betatron oscillation) is slightly different from the natural betatron oscillation. Although this difference is usually ignored because it becomes smaller when driving frequency and betatron frequency gets closer, it has more than 6% of impact on lattice function measurement in typical operations of an AC dipole. This paper shows such difference can be seen in measurements using the AC dipole of the Tevatron and explained as difference of lattice functions between the natural and driven betatron oscillation. The paper also mentions lattice function measurements based on the AC dipole.

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I. INTRODUCTION

A. AC Dipole

Advantages of the AC dipole over a conventional kicker or pinger magnet are, when the field strength of the dipole is adiabatically ramped up and ramped down, it can create large oscillations without decoherence and emittace growth [**?**]. These properties of the AC dipole makes it useful to measure both linear and nonlinear parameters of a synchrotron, particularly a hadron synchrotron because it allows measurements without interfering its normal operations. The AC dipole has been used and tested in mainly BNL RHIC and also CERN SPS [**?**]. Now, it has been used in FNAL Tevatron to study its optics and preparation is underway for LHC too [**?**].

B. Two Driving Terms of the AC Dipole

As details are shown in the next section, the driven betatron oscillation can be written in the same form as the natural betatron oscillation but its lattice functions are different. The difference arises because the driving force is localized and two driving terms are created as a consequence. For instance, frequencies of the beam revolution and AC dipole in the Tevatron are $f_r \simeq 47.7$ and $f_d \simeq 20.5$ kHz and a tune of the AC dipole is $\nu_+ \equiv f_d/f_r \simeq 0.43$. As there are two peaks observed at ν and $1-\nu$ in Schottky monitors, where ν is the fractional part of betatron tune, the circulating beam sees not only ν_+ but also see $\nu_- \equiv 1-\nu_+ = 0.57$ as a tune of the AC dipole (Fig 1). This way, a local driving force like an AC dipole creates a pair of driving terms at $\nu_\pm$. In this paper, the driving term closer to ν is called the primary and the other is called the secondary. In the following, it is convenient to define tune of the primary driving term:

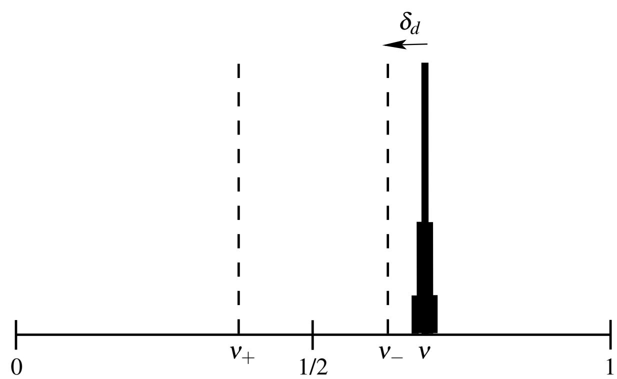

FIG. 1: Tune spectrum (vertical) on the frame moving with the beam. An AC dipole creates two driving terms at $\nu_\pm$ (dashed lines). For the Tevatron AC dipole, ν_- is closer to ν and the driving tune $\nu_d = \nu_-$. δ_d is distance from ν_d to ν. In typical operations of an AC dipole, ν_d is outside of the tune spread (black part).

$$\nu_d \equiv \begin{cases} \nu_+ & \text{when } |\nu_+ - \nu| < |\nu_- - \nu| \\ \nu_- & \text{when } |\nu_- - \nu| < |\nu_+ - \nu| \, . \end{cases} \tag{1}$$

For the Tevatron AC dipole, $\nu_d = \nu_- = 0.57$ and $1-\nu_d = \nu_+ = 0.43$.

$\delta_d \equiv \nu_d - \nu$ is a parameter to describe the distance between ν_d and ν. Ideally, in the limit of $\delta_d \to 0$, the effect of the primary driving term becomes dominant and the secondary driving term can be simply ignored but in reality, for hadron synchrotrons such as Tevatron, RHIC, and LHC, $|\delta_d|$ is in the order of 0.01 at minimum and it is not always small enough to ignore the secondary driving term.

The next section shows the driven betatron oscillation can expressed in the same form as the natural betatron oscillation even when the secondary driving term is included and their difference can be expressed as difference of their lattice functions. Section III shows the effects of the secondary driving term can be actually seen in measurements using the AC dipole in the Tevatron and they can explained as the difference of lattice functions.

II. DRIVEN BETATRON OSCILLATION

A. Notations and Coordinate System

When analyzing turn-by-turn (TBT) data of the driven betatron oscillation, it is convenient to use transverse position and angle at one location of a synchrotron $\bar{s}$ ($0 \leq \bar{s} < C$) n revolutions after the AC dipole is turned on:

$$x_n(\bar{s}) \equiv x(nC+\bar{s}) \tag{2}$$

$$x'_n(\bar{s}) \equiv x'(nC+\bar{s}) \ , \tag{3}$$

where C is the circumference of the machine and $\bar{s}$ is introduced to avoid confusion with longitudinal coordinate s ($-\infty < s < \infty$). The point of reference $\bar{s}=0$ is chosen to be the location of the AC dipole. Let $\psi(\bar{s})$ be the phase advance of the natural betatron oscillation measured from the location of the AC dipole ($\bar{s}=0$) to $\bar{s}$:

$$\psi(\bar{s}) \equiv \int_0^{\bar{s}} \frac{ds}{\beta(s)} \ . \tag{4}$$

Obvious $\bar{s}$ dependences of parameters are sometimes ignored in the following.

B. Magnitudes of Two Driving Terms

As discussed in the previous section, field strength of an AC dipole is adiabatically ramped up and ramped down before and after the measurement. When the AC dipole is on its flat top, position of the driven betatron oscillation $x_{d,n}$ is given by [?]

$$\begin{aligned} x_{d,n}(\bar{s}) \simeq\ & \frac{\theta_d}{4\sin(\pi(\nu_+-\nu))}\sqrt{\beta(0)\beta} \\ & \times\cos(2\pi\nu_+ n+\psi+\pi(\nu_+-\nu)+\chi_d) \\ & +\frac{\theta_d}{4\sin(\pi(\nu_--\nu))}\sqrt{\beta(0)\beta} \\ & \times\cos(2\pi\nu_- n+\psi+\pi(\nu_--\nu)-\chi_d)) \ , \end{aligned} \tag{5}$$

where θ_d is the maximum kick angle of the AC dipole defined with the maximum magnetic field B_d, length of the dipole ℓ_d, and magnetic rigidity $(B\rho)$

$$\theta_d \equiv \frac{B_d \ell_d}{(B\rho)} \ , \tag{6}$$

χ_d is the initial phase of the AC dipole (the phase when the beam is first kicked by the AC dipole), and $\beta(0)$ is β at the location of the AC dipole. As seen in the equation, two terms are completely symmetric and each term represents the effect from driving terms at $\nu_\pm$. The exact expression of $x_{d,n}(\bar{s})$ includes terms inversely proportional to ramp times but they are typically less than 1% of the largest term when the AC dipole is slowly ramped up and down.

Since δ_d is small (typically the order of 0.01) in usual operations of an AC dipole, one term in Eq (5) is much larger than the other and the smaller term has been simply ignored in data analyses. To estimate the effect of the smaller driving term, define ratio of two terms in Eq (5):

$$\begin{aligned} \lambda_d(\delta_d) &\equiv \begin{cases} \sin(\pi(\nu_+-\nu))/\sin(\pi(\nu_--\nu)) \text{ when } \nu_d=\nu_+ \\ \sin(\pi(\nu_--\nu))/\sin(\pi(\nu_+-\nu)) \text{ when } \nu_d=\nu_- \end{cases} \\ &= \frac{\sin(\pi\delta_d)}{\sin(2\pi\nu+\pi\delta_d)} \simeq \frac{\pi\delta_d}{\sin(2\pi\nu)} \ . \end{aligned} \tag{7}$$

For hadron storage rings such as Tevatron, RHIC, and LHC, δ_d is desired to be $|\delta_d| \lesssim 0.01$. Since the tune of the Tevatron is $\nu \simeq 0.58$, $|\lambda_d| \simeq 6.5\%$ when $|\delta_d|=0.01$. If $\nu \simeq 0.3$ such as RHIC (polarized proton operations) and LHC, the separation between $\nu_\pm$ gets larger and $|\lambda_d|$ becomes smaller but it is still about 3% when $|\delta_d|=0.01$. Notice these are effects on amplitude. Since β function is proportional to square of amplitude, if it is measured by simply ignoring term of the secondary driving term, error in the measurement is 13% for the Tevatron and 6% for RHIC and LHC when $|\delta_d|=0.01$. These effects of the secondary driving term could be seen in recent measurements using the AC dipole of the Tevatron. They are presented in section III.

C. Lattice Functions

By using formulae of trigonometric functions, without ignoring the smaller term, Eq (5) can be written in a similar form of the natural betatron oscillation:

$$x_{d,n}(\bar{s},\delta_d) = A_d\sqrt{\beta_d}\cos(2\pi\nu_d n+\psi_d\pm\chi_d) \ , \tag{8}$$

where A_d is a quantity with dimensions of (length)$^{1/2}$:

$$A_d(\delta_d) \equiv \frac{\theta_d}{4\sin(\pi\delta_d)}\sqrt{(1-\lambda_d^2)\beta(0)} \ , \tag{9}$$

β_d is amplitude function of the driven betatron oscillation:

$$\beta_d(\bar{s};\delta_d) \equiv \frac{1+\lambda_d^2-2\lambda_d\cos(2\psi-2\pi\nu)}{1-\lambda_d^2}\beta \ , \tag{10}$$

ψ_d is the phase advance of the driven betatron oscillation measured from the location of the AC dipole:

$$\psi_d(\bar{s};\delta_d) \equiv \int_0^{\bar{s}} \frac{ds}{\beta_d(s;\delta_d)} \ , \tag{11}$$

and the sign in front of χ_d is positive when $\nu_d=\nu_+$ and negative when $\nu_d=\nu_-$. A relation between ψ and ψ_d is given by

$$\tan(\psi_d - \pi\nu_d) = \frac{1+\lambda_d}{1-\lambda_d}\tan(\psi-\pi\nu) = \frac{\tan(\pi\nu_d)}{\tan(\pi\nu)}\tan(\psi-\pi\nu)\ . \quad (12)$$

Parameters corresponding to α and γ can be also defined as the natural betatron oscillation:

$$\alpha_d(\bar{s};\delta_d) \equiv -\frac{1}{2}\frac{d\beta_d}{ds} = \frac{1+\lambda_d^2-2\lambda_d\cos(2\psi-2\pi\nu)}{1-\lambda_d^2}\alpha - \frac{2\lambda_d\sin(2\psi-2\pi\nu)}{1-\lambda_d^2} \quad (13)$$

$$\gamma_d(\bar{s};\delta_d) \equiv \frac{1+\alpha_d^2}{\beta_d} = \frac{1+\lambda_d^2+2\lambda_d\cos(2\psi+2\arctan\alpha-2\pi\nu)}{1-\lambda_d^2}\gamma\ . \quad (14)$$

When A_d, β_d, α_d, and γ_d are defined this way, a relation like Courant-Snyder invariance also holds:

$$A_d^2 = \gamma_d x_{d,n}^2 + 2\alpha_d x_{d,n}x'_{d,n} + \beta_d x'^2_{d,n}\ . \quad (15)$$

From Eqs (10), (13), (14), and (11), lattice functions of the driven betatron oscillation β_d, α_d, γ_d, and ψ_d are functions of not only $\bar{s}$ but also δ_d and their differences from β, α, γ, and ψ are the factor of $2\lambda_d$ in the leading order (except for α_d since there is an extra term). Although these differences vanish in the limit of $\delta_d \to 0$ (and so $\lambda_d \to 0$), $|\delta_d| \simeq 0.01$ is the practical limit and β_d, α_d, γ_d, and ψ_d all differ from β, α, γ, and ψ about 13% in the Tevatron and 6% in LHC and RHIC (maybe more for α_d). Also notice β_d, α_d, and γ_d depend on $\bar{s}$ through not only β, α, and γ but also ψ.

III. PROPERTIES OF THE DRIVEN BETATRON OSCILLATION

A. Phase Space Trajectory

In three locations of the Tevatron, A0, B0, and D0, there are spaces with no magnet (except solenoids of detectors for B0 and D0) between two BPMs. Since the beam runs on a straight line in such spaces, position and angle at any location between two BPMs can determined without using lattice information.

From Eq (15), TBT position and angle of the driven betatron oscillation also form an ellipse at a location of a synchrotron. Since not only A_d but also β_d, α_d, and γ_d depend on δ_d, both area and shape changes with δ_d for a phase space ellipse of the driven betatron oscillation. Fig 2 shows the measured phase ellipses of the driven betatron oscillation when $\delta_d = -0.04$, -0.02, and -0.01 magnetic field of the AC dipole is kept the same. Measurements were done

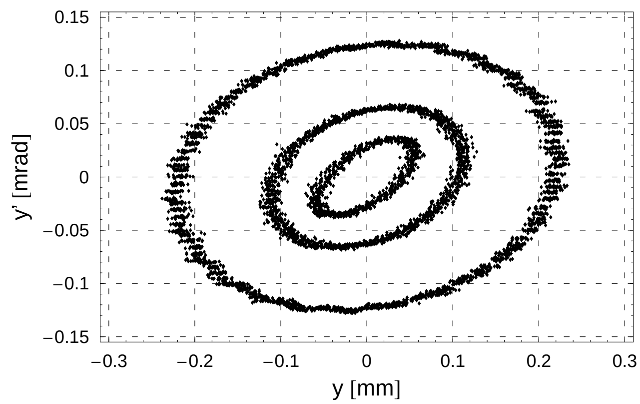

FIG. 2: Phase space (vertical) trajectories of the driven betatron oscillations when δ_d is -0.01 (outermost), -0.02 (middle), and -0.04 (innermost) while the field strength is kept the same. The location is D0 interaction point where α is zero by design. Area gets larger and tilt gets smaller when $|\delta_d|$ gets smaller.

with the same proton beam with small momentum spread of $\Delta p/p \simeq 3.5\,10^{-4}$ at the injection energy 150 GeV. The location is the D0 interaction point of the Tevatron where $\alpha \simeq 0$ by design. In the figure, it is possible to observe tile angle gets larger and the shape gets thinner when δ_d decreases.

By fitting Eq (15) to data points of an ellipse in Fig 2, its area πA_d^2, which is invariant of the longitudinal position, and β_d, α_d, and γ_d at the location can be determined. Fig 3 shows the measured relation between α_d and δ_d where each data point is determined from the fit to the ellipse. Again, the location is the D0 interaction point. The line in Fig 3 is the fit of Eq (13) to data points with fit parameters of α and ψ. Since α_d describes the correlation between position and angle, the fact $|\alpha_d|$ gets larger with $|\delta_d|$ corresponds to the tile angle increasing with $|\delta_d|$ in Fig 2. $\alpha = \alpha_d(\delta_d = 0)$ can be determined from this fit and it is about -0.015 in this case. Notice the difference between α and α_d is as large as 100% even when $\delta_d = -0.01$. The reason of this large difference comes from the second term of Eq (13). Although the effect of this extra term is negligible when $\alpha > 2\lambda_d$, it makes large difference when $\alpha \simeq 0$, hence at waists of low β insertions, because the maximum of $2\lambda_d$ is added to α depending on the phase advance. Since β_d does not have such an extra term, difference between β_d and β is the factor of $2\lambda_d$. Difference between β_d and β is discussed in the next section.

B. Amplitude Function β_d

From Eqs (9) and (10), the amplitude of the driven betatron oscillation at $\bar{s}$ is given by

$$A_d\sqrt{\beta_d(\bar{s})} = \frac{\theta_d\sqrt{\beta(0)\beta(\bar{s})}}{4\sin(\pi(\nu_d-\nu))}\sqrt{1+\lambda_d^2-2\lambda_d\cos(2\psi(\bar{s})-2\pi\nu)}\ . \quad (16)$$

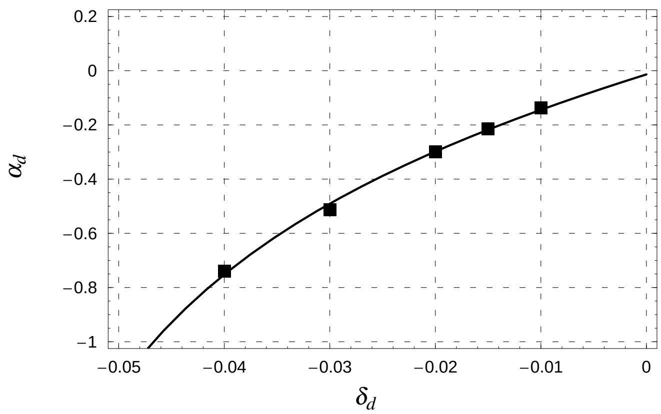

FIG. 3: Relation between α_d and δ_d at D0 interaction point. The squares are data points and the line is the fit of Eq (13). $\alpha = \alpha_d(0) \simeq -0.015$.

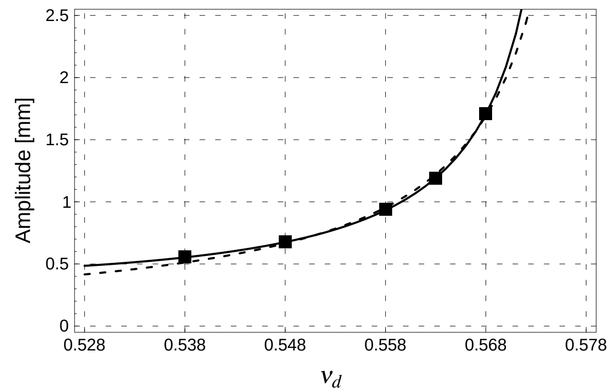

FIG. 4: Relation between the oscillation amplitude of the driven betatron oscillation and ν_d at one BPM in the Tevatron (section B21). The squares represent the data points and the solid and dashed lines are fits with and without the effect of the secondary driving term included.

Notice $\bar{s}$ dependence of the amplitude is determined by not only β but also the factor $[1+\lambda_d^2-2\lambda_d\cos(2\psi-2\pi\nu)]$ through phase ψ. Since this factor has the same phase dependence as β beat [**?**], the effect of the secondary driving term cannot be distinguished from the β beat. Ignoring the secondary driving term is the same as taking the limit of $\lambda_d \to 0$. Then, the amplitude becomes

$$A_d\sqrt{\beta_d(\bar{s})} \xrightarrow{\lambda_d \to 0} \frac{\theta_d\sqrt{\beta(0)\beta(\bar{s})}}{4\sin(\pi(\nu_d-\nu))} \; . \tag{17}$$

In the similar data sets as the previous section, it is possible to see the effect of the secondary driving term described by Eq (16). Fig 4 shows amplitudes of the driven betatron oscillation measured at one BPM in the Tevatron. Five data points represent amplitudes when $\nu_d = 0.568$, 0.563, 0.558, 0.548, and 0.538 and the same magnetic field of the AC dipole. Other conditions are the same as the measurements in the previous section. Since the tune was set to the nominal value 20.578 prior to the measurement, δ_d for each data points are roughly -0.01, -0.015, -0.02, -0.03, and -0.04. The solid and dashed lines are the fit of Eqs (16) and (17) to these data sets. The fit parameters are ν, $\frac{1}{4}\theta_d[\beta(0)\beta(\bar{s})]^{1/2}$, and ψ for Eq (16) and ν and $\frac{1}{4}\theta_d[\beta(0)\beta(\bar{s})]^{1/2}$ for Eq (17). Although, from comparison of two fits at one BPM, it is not clear that the fit including the secondary driving term is better or not, it becomes clearer when fit parameters of all BPMs are compared.

Fig 5 shows tunes determined at each BPM location by repeating the same fit as Fig 4. The solid and dashed lines represent fits of Eqs (16) and (17). Since tune ν is a constant around the ring, variations of measured tunes over BPMs give a sence of inaccuracy. Fig 5 clearly indicates the model including the secondary driving gives better description to data.

For the natural betatron oscillation, square of the amplitude at one location of a synchrotron is given with a constant of motion A by $A^2\beta$. If there is a pair of BPMs in a drifting space of the ring, like the case of the Tevatron, A^2 can be determined from the phase space area. This is a typical process to determine (uncoupled) β functions at BPMs from TBT data of the natural betatron oscillation. If the precess is repeated for TBT data of the driven betatron oscillation, from Eq (**??**), what is measured is β_d instead of β. In general, measuring β from TBT data of the driven betatron oscillation requires more than one data set and actually the fits of the previous paragraph can be used. From the fits, tune ν and a parameter $\frac{1}{4}\theta_d[\beta(0)\beta(\bar{s})]^{1/2}$ at BPMs are given. A_d is also determined from the phase space area. When tune and A_d are known, the constant $\frac{1}{4}\theta_d\beta(0)^{1/2}$ in front of $\beta(\bar{s})^{1/2}$ is given by

$$\frac{1}{4}\theta_d\sqrt{\beta(0)} = \frac{A_d\sin(\pi\delta_d)}{\sqrt{1-\lambda_d}} \; . \tag{18}$$

This way, β at BPMs can be determined from multiple TBT data sets of the driven betatron oscillation.

Fig 6 is a comparison between vertical β and β_d when $\delta_d = -0.01$. They are both measured from TBT data of the driven betatron oscillation as described in the previous para-

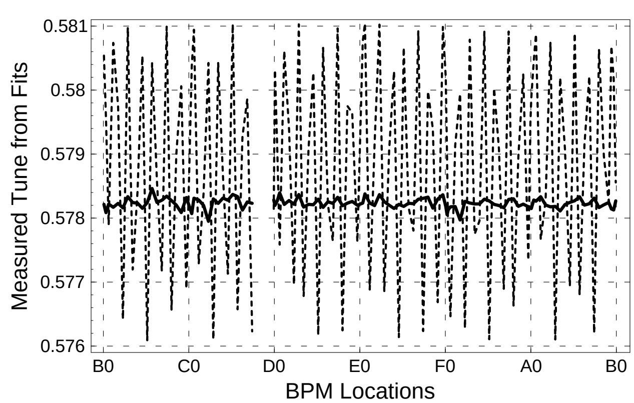

FIG. 5: Vertical tune measured with each BPM from the fits to amplitudes. The solid and dashed lines are from fits with and without the effect of the secondary driving term included. The tune was adjusted to its nominal value 20.578 before the measurement.

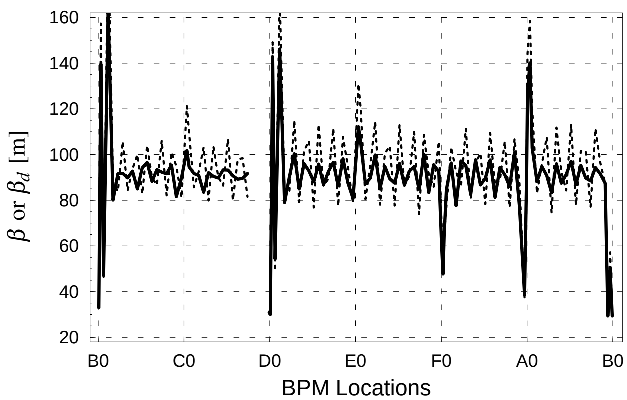

FIG. 6: Comparison of vertical β (solid) and β_d when $\delta_d = -0.01$ (dashed) at vertical BPMs in the Tevatron. Both of them are measured from the same data set(s). Depending on δ_d. β_d is expected to show β beta like behavior with amplitude of 13%.

graph. Notice the average value of β is about 100 m and the dashed line (β_d) is showing beating of roughly $\pm 10 - 15$ m compared to the solid line (β) as expected in II C. The graph indicates the difference between β and β_d created by the secondary driving term is still noticeable even when $|\delta_d| = 0.01$.

The last fit parameter of Fig 4 is the phase advance of the natural betatron oscillation ψ. Therefore, the phase can be also determined from the fit of the amplitudes. Notice, since phase advance is also different between the natural and driven betatron oscillation, Eq (11) or (12), ψ cannot be directly measured from phase difference of driven betatron oscillation at BPMs.

IV. CONCLUSION

When a beam is driven by an AC dipole, its motion is governed by lattice functions β_d, α_d, γ_d, and ψ_d. They satisfy the similar relations as the lattice functions of the natural betatron oscillation. Their differences are in the order of $2\lambda_d$ (except α and α_d) and created by the secondary driving term. The paper showed such differences are observed as expected when the beam is excited by the AC dipole in the Tevatron. It was also discussed how to measure lattice functions of the natural betatron oscillation from TBT data sets of the driven oscillation without ignoring the difference in their lattice functions.

Acknowledgments

Authors would like to thank

[1] M. J. Syphers, Fermilab Report, Beams-doc-1088, (2004).